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CONTINUITY, DETERMINISM, AND REALITY

BY

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Introduction.

The following considerations are an attempt to discuss the ancient and time honoured metaphysical concepts of continuity, determinism, and reality with the help of a simple, almost trivial example. Theoretical physics has, by its own efforts, come to a point where it had to abandon a great deal of traditional philosophical ideas and to replace them by new ones. But there are still leading physicists, amongst them EINSTEIN (1), DE BROGLIE (2), and SCHRÖDINGER (3), who have not accepted the new way of thinking. Therefore, a careful analysis of the philosophical situation in physics seems not to be superfluous. EINSTEIN himself has formulated on several occasions his objections against the current interpretation of quantum mechanics not in obscure philosophical terms, but with the help of simple models. The same method will be followed here; in fact, the model discussed is actually due to EINSTEIN (4). It makes it possible to illustrate abstract philosophical ideas by elementary geometrical considerations; these provide of course no direct answer to the metaphysical problems, but reduce them to clearly distinct alternatives and help thus to clarify the logical situation.

Part I. General Considerations.

1. Continuity.

I maintain that the mathematical concept of a point in a continuum has no direct physical significance. It has, for instance, no meaning to say the value of the coordinate x of a mass-point, or of the centre of mass of an extended body, has a value represented in a given unit by a real number (like $x = \sqrt{2}$ inch. or $x = \pi$ *cm.*).

Modern physics has achieved its greatest successes by applying the methodological principle that concepts which refer to distinctions beyond possible experience have no physical meaning and ought to be eliminated. This principle was certainly operative in many instances since Newton's time. The most glaringly successful cases are Einstein's foundation of special relativity based on the rejection of the concept of aether as a substance absolutely at rest, and Heisenberg's foundation of quantum mechanics based on the elimination of orbital radii and frequencies of electronic structures in atoms. I think that this principle should be applied also to the idea of physical continuity. Now consider, for instance, a statement like $x = \pi cm.$; if π_n is the approximation of π by its first n decimals, then the differences $\pi_n - \pi_m$ are, for sufficiently large n and m, smaller than the accuracy of any possible measurement (even if it is conceded that this accuracy may be indefinitely improved in the course of time). Hence, statements of this kind should be eliminated.

That does not mean that I reject the mathematical concept of real number. It is indispensible for applying analysis. The situation demands a description of haziness of physical quantities with the help of real numbers.

The proper tool for this is the concept of probability. It can be assumed that sentences like the following have a meaning: The probability for the value of a physical quantity to be in a given interval (represented by two real numbers) has a certain value (again a real number). Or, with other words, for any quantity x there exists a probability density $P(x)$.

This attitude is generally accepted in quantum mechanics. But it has actually a more fundamental significance and is only indirectly connected to the special features characteristic of quantum mechanics. It ought to be applied to classical mechanics as well.

2. Determinism.

Classical mechanics has its roots, since Newton's time, in astronomy where the prediction of constellations was its main aim. Thus, the deterministic character of the mechanical laws is stressed in the traditional presentations. When mechanics is applied to micro-phenomena, it is, however, necessary to analyse the meaning of the term determinism a little deeper. The mechanical laws have the property that a precisely given initial state (configuration and velocities) determines at any time a sharp final state. There are two possibilities: Either a small change of the parameters in the initial state (small compared with the total range) produces only small changes of the final values for all times; then the orbit defined by the initial conditions is *stable*. Or this is not the case, the final deviations increase in time beyond any limit; then the orbit is *instable*.

In astronomy, much work has been done to prove the stability of the planetary system. For our purpose, the results of these investigations are irrelevant. What matters is that there exist simple mechanical systems of a type familiar in atomic physics (kinetic theory of gases) for which all orbits are instable. These systems display therefore only what I should call *weak determinism;* the future state can be predicted only if the initial state is defined absolutely sharply, in the sense of the mathematical concept of a point in a continuum; the slightest initial deviation produces an ever increasing vagueness of the final state. Thus, for systems of this kind, there is a close connection between the problems of continuity and determinism. If the point in a continuum has no physical meaning, it is impossible to maintain that systems of this type behave in a deterministically predictable way. Hence, for a wide class of mechanical systems, the traditional form of (classical) mechanics ought to be replaced by a statistical method which uses right from the beginning the notion of probability: There exists, for any coordinate x, velocity v, and any instant, a probability density $P(x, v, t)$.

The simplest example of this type of systems is the model, suggested by EINSTEIN with a very different intention, namely, to demonstrate the incompleteness of quantum mechanics (a question to which I shall return presently). It is the model of a one-dimensional one-particle gas and consists of a mass-point moving in a straight line (coordinate x) up and down between two points ($x = 0$ and $x = l$) where it is elastically reflected[1]. In a diagramme, the motion is represented by a zig-zag line

[1] If the assumption of an extensionless mass-point and perfect elasticity seems to be too unrealistic, one may take the centre of mass of a finite body running against high and steep potential walls at $x = 0$ and $x = l$.

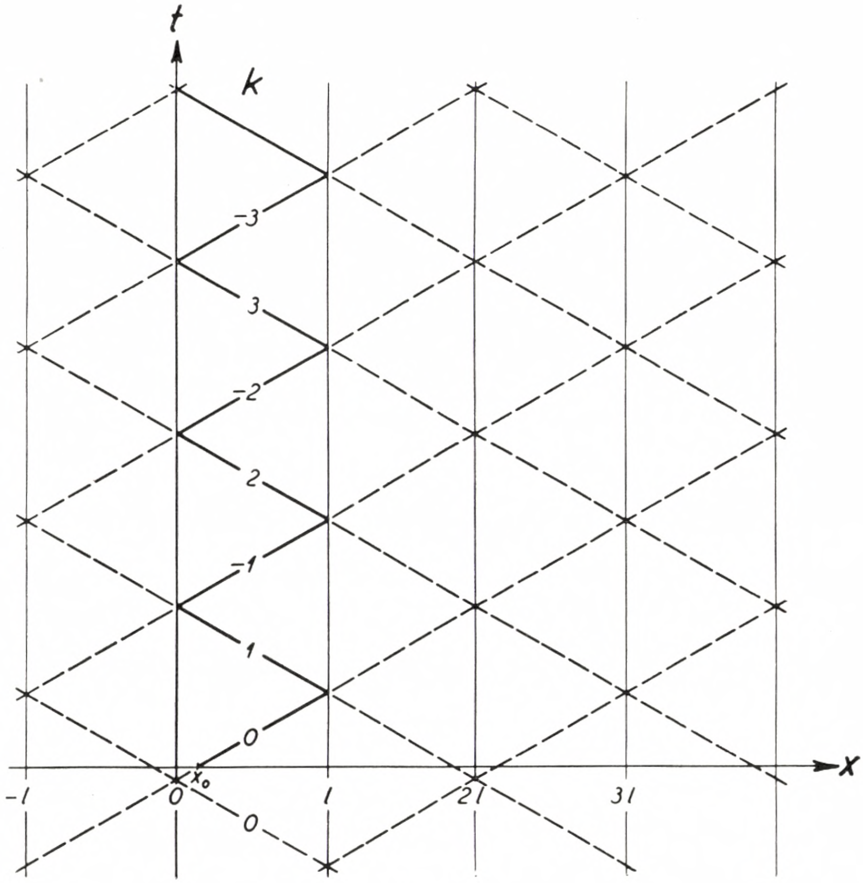

Fig. 1.

inside the strip $0 < x < l$ with alternating constant inclinations $\pm v_0$, where v_0 is the initial velocity. By taking successive images of this figure at the boundary lines of the strip, the diagramme Fig. 1 is obtained which is symmetric at vertical lines $x = kl$ $(k = 0, \pm 1, \pm 2, \ldots.)$ and has the period $2\,l$. The zig-zag motion is therefore equivalent to two sets of parallel, synchronized straight line motion. It is obvious that $x(t)$ is, for any t, determined by $x_0 = x(0)$ and v_0.

But, if x_0, v_0 are changed by Δx_0, Δv_0, the diagramme of Fig. 2 is obtained, which illustrates that Δx increases proportionally to $t, \Delta x = \pm t \Delta v_0$. After the time $t_c = l/\Delta v_0$, the variation of x is larger than the whole range l of x. Hence, the system is

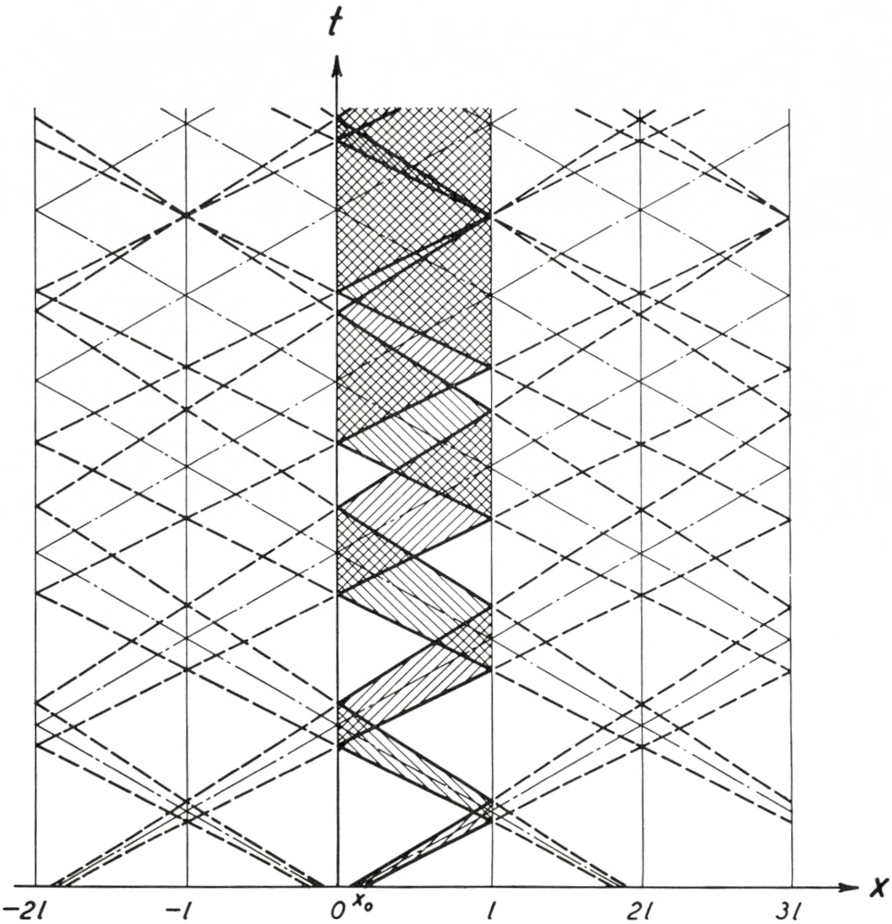

Fig. 2.

perfectly instable and behaves, for $t > t_c$, in an indeterministic manner.

Though this is perfectly trivial, I have never seen it pointed out[1].

3. Reality.

The question what we mean by the expression "physical reality" is closely connected with the previous considerations on

[1] For an unbounded straight line motion, the question of stability has no meaning as there is no range (like l in the Einstein model) with which to compare $\Delta x(t)$. The usual considerations on mechanical determinism miss this essential point of a final range.

continuity and determinism. EINSTEIN, in the paper quoted (4), describes "the programme that, until the introduction of quantum mechanics, was unquestionably accepted for the development of physical thinking" in the following way (translated from the original German): "Everything is to be reduced to conceptual objects situated in space-time and to strict relations which hold for these objects. In this description, nothing appears which refers to empirical knowledge about these objects. A spatial position (relative to the co-ordinate system used) is attributed to, say, the moon at any definite time, quite independently of the question whether observations of this position are made or not. This kind of description is meant if one speaks of the physical description of a "real external world". . . . . EINSTEIN then discusses the question whether quantum mechanics leads to a description of the behaviour of macro-bodies, which corresponds to this notion of reality, and his answer is no. He considers the model of a one-dimensional one-particle gas (discussed above) and compares the classical motion with fairly sharp initial position and velocity with a special solution of the Schrödinger equation

$$\psi = Ae^{iat} \sin bx = \frac{1}{2i} Ae^{i(at+bx)} - \frac{1}{2i} Ae^{i(at-bx)}, \tag{1}$$

(a and b being properly chosen constants); this represents a state where the momentum has either of two opposite equal values and the probability of position is, for sufficiently high momentum, constant apart from small periodic variations. He continues (translated): "For a macro-system we are sure that it is at any time in a 'real state' which is correctly described with good approximation by classical mechanics. The individual macro-system of the kind considered by us has therefore at any time an almost sharply defined coordinate (of its centre of mass)—at least if averaged over a small interval of time—and an almost sharply defined momentum (defined also in regard to sign). None of these results can be obtained from the ψ-function. It contains only such statements which refer to a *statistical ensemble* of the kind considered". And a few lines later he concludes: "Quantum mechanics describes ensembles of systems, not individual systems. The description with the help of a ψ-function is thus an *incomplete* description of a single system, not a description of its 'real state'."

This consideration, as it stands, is not conclusive, as the function ψ chosen by EINSTEIN is a very special solution of the wave equation, not adapted to the initial conditions and therefore not suited to illuminate the question whether quantum mechanics is able to describe the individual macro-body in a "realistic" manner—like classical mechanics—or can tackle only statistical ensembles. This question will be treated in some detail in the second part of this paper. Here another point must be discussed, which is implicitly contained in Einstein's publication and obviously foremost in his mind[1].

In the previous sections, it has been shown that no physical meaning can be attributed to a sharp value of a co-ordinate and that therefore the description of a position in Einstein's model should be given in a hazy but realistic manner through a probability density $P(x)$; that, further, the laws of classical mechanics should be formulated not in terms of orbits, but of a time-dependent probability density $P(x, v, t)$. If this is done, classical mechanics is actually not dealing with a single system, but with a statistical ensemble, and Einstein's criticism of quantum mechanics, quoted above, taken literally, fails as it would apply in the same way to the classical theory. However, what EINSTEIN really means, is evident from another sentence of his article which reads (translated): "The fact that, for the macro-system considered, not every function ψ satisfying the Schrödinger equation corresponds approximately to a description of a real phenomenon in the sense of classical mechanics, is particularly obvious by considering a ψ-function which is formed by the superposition of two functions of the type (1) whose frequencies (energies) are essentially different. For, to such a superposition, there is no corresponding 'real case' of classical mechanics (still, however, a statistical ensemble of such 'real cases' according to Born's statistical interpretation)."

Classical mechanics, formulated statistically as it ought to be, is still a "description of reality" according to Einstein's definition,

[1] I have to thank Professor W. PAULI for giving me, in some letters, an explanation of Einstein's ideas, obtained in oral discussions at Princeton, and his own comments.

as one can *think* the single, sharp state as existing (though not observe it with the accuracy demanded by the mathematical concept of sharpness) and then obtain the physical vagueness by applying the ordinary laws of probability. For instance, one can think of a particle in a straight line being at x_1 and then the physical situation "we know that it is near x_1" by a probability density $p(x - x_1)$ (where the function $p(x)$ is different from zero only near $x = 0$). If we only know that the particle is either near x_1 or near x_2, the probability density will be

$$P(x) = a_1 p(x - x_1) + a_2\, p(x - x_2),\ a_1 + a_2 = 1, \tag{2}$$

according to the ordinary rules of probability calculus.

In quantum mechanics the situation, however, is different. If $\varphi(x - x_1)$ is the Schrödinger function describing a particle being near x_1, the probability density is $p(x - x_1) = |\varphi(x - x_1)|^2$. If we know that the particle is either near x_1 or near x_2, the situation is described by the Schrödinger function $\psi(x) = c_1\varphi(x - x_1) + c_2\varphi(x - x_2)$ and the resultant probability is

$$\left.\begin{aligned} P(x) = |\psi(x)|^2 = a_1 p(x - x_1) + a_2\, p(x - x_2) + J(x), \\ a_1 = |c_1|^2,\ a_2 = |c_2|^2, \end{aligned}\right\} \tag{3}$$

where the additional term

$$J(x) = c_1 c_2^* \varphi(x - x_1)\varphi^*(x - x_2) + c_1^* c_2 \varphi^*(x - x_1)\,\varphi(x - x_2) \tag{4}$$

represents the "interference of probabilities". It has no classical analogue; even if it is practically negligible for $t = 0$, it may become appreciable for certain x-values at later instances.

The existence of this interference phenomenon excludes the possibility to think of the particle as having a definite position (and velocity) at any instant and to connect these positions in imagination to an orbit, and this is the reason why EINSTEIN declares quantum mechanics to be incomplete[1]. He insists that, at least for macro-bodies, a theory cannot be regarded as satisfactory unless it conforms with his idea of reality.

[1] EINSTEIN discusses in this connection the ideas of DE BROGLIE, BOHM, SCHRÖDINGER a. o. who tried, in different ways, to interpret the formalism of quantum mechanics in terms of classical concepts, but he rejects these attempts as unsatisfactory.

This is a philosophical creed which can be neither proved nor disproved by physical arguments. But what can be done is this: one can formulate another concept of physical reality which takes account of the actual existence of the interference phenomenon in the atomistic region and goes over into the traditional one (that accepted by EINSTEIN) for macro-bodies. This I have done in a systematic, but rather abstract way, at another place (5). I shall not repeat these considerations here, but illustrate them only with the help of the model used above, a particle oscillating on a line between two elastically reflecting boundaries.

The main point is that the physicist has not to do with what can be thought of (or imagined), but what can be observed. From this standpoint a state of a system at a time t, when no observation is made, is not an object of consideration. But as soon as an observation is made, the situation found has to be regarded as the final state of the phenomenon defined by a previously observed initial state and, if future observations are envisaged, also as the initial state of the further development. This "reduction of probability" is not characteristic of quantum mechanics, but has also to be applied to classical mechanics if it is formulated in terms of probability: Any observation for checking a predicted probability density "destroys" it and produces a new one which has to serve as initial state for further predictions.

But from this standpoint the interference phenomenon looses much of its paradoxial character. For the one-dimensional model, an actual observation determines not the complex amplitudes $c_1 = \sqrt{a_1}e^{i\alpha_1}$, $c_2 = \sqrt{a_2}e^{i\alpha_2}$, but only the probabilities (relative frequencies) $a_1 = |c_1|^2$, $a_2 = |c_2|^2$; the phases α_1, α_2 remain entirely unknown and undetermined, and the interference term vanishes if averaged over the phase difference $\alpha_1 - \alpha_2$. For more complicated systems (like the optical interferometers), the distribution in the final state may of course show interference fringes, which classical theory cannot explain; but this appears only paradoxial from the traditional (Einstein's) standpoint where a non-observed intermediate state is declared to be just as real as an actually observed final state.

The situation can be illustrated by a detailed discussion of our model. This will be done in the second part of this paper.

Part II. Mathematical Considerations.

The model which will now be investigated in more detail seems to be the simplest mechanical system with a finite range of the variables (co-ordinate, velocity) for which the exact solution can be found. The Hamiltonian has essentially only a kinetic part; the potential energy due to reflection at the boundaries can be replaced by certain periodicity conditions, and the equations of motion then can be solved, in the classical and quantum treatment as well, with the help of Kelvin's method of images. The resulting formulae are simple and well suited for a discussion of several important problems, as the transition from the initial individualistic to the final statistical description, the characteristic distinctions of classical and quantum treatment, the reduction of probability through observation, and the interference of probabilities.

1. Classical treatment of the one-particle one-dimensional gas.

The orbit of a particle in Einstein's model, starting at $t = 0$ from the point $x = x_0$ with the velocity $v = v_0$, is analytically given by

$$\left\{\begin{aligned} x &= 2lk - x_0 - v_0 t, \qquad & t_{2k-1} \leqslant t \leqslant t_{2k}, \\ x &= -2lk + x_0 + v_0 t, \qquad & t_{2k} \leqslant t \leqslant t_{2k+1}, \end{aligned}\right\} \tag{1.1}$$

where

$$t_k = \frac{kl - x_0}{v_0}, \quad k = 0, \pm 1, \pm 2, \ldots \quad . \tag{1.2}$$

It is convenient (as already indicated in Fig. 1) to replace the one-particle system by a periodic system, consisting of an infinite number of synchronized particles, by dropping the conditions $t \geqslant 0$, $0 \leqslant x \leqslant l$ (silently assumed in (1.1)). This procedure will be denoted by the short name "periodic continuation". According to the programme explained in Part I, the "deterministic" description (1.1), (1.2) shall be replaced by a statistical one, with the help of a probability density, $P(x, v, t)$. We have to do with a case of statistical mechanics where the system is not in statistical equilibrium, but develops in time from a given initial distribution $P(x, v, 0)$. The only condition for $P(x, v, t)$ is that which expresses the conservation of probability; it follows from Lionville's theorem,

$$\frac{\partial P}{\partial t} + [P, H] = 0, \tag{1.3}$$

where $H(x, p)$ is the Hamiltonian as function of coordinate and momentum and

$$[P, H] = \frac{\partial P}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial H}{\partial x} \tag{1.4}$$

the Poisson bracket.

H consists of the kinetic energy $p^2/2m$, and the potential energy representing the reflective power of the walls. As this force is assumed to be infinitely strong, it can be replaced by certain periodicity conditions which will be derived presently. With $H = p^2/2m$ and $p = mv$, (1.4) reduces to

$$\frac{\partial P}{\partial t} + v\frac{\partial P}{\partial x} = 0. \tag{1.5}$$

The periodicity conditions follow from the consideration that the solution must have the same value at a given point x (in $0 \leqslant x \leqslant l$) after each reflection; for instance, after one reflection at $x = 0$, one has

$$P(x, v, t) = P\left(x, -v, t - \frac{2x}{v}\right) \tag{1.6a}$$

and, after two reflections at $x = 0$ and $x = l$,

$$P(x, v, t) = P\left(x, v, t + \frac{2l}{v}\right). \tag{1.6b}$$

The general solution of (1.5) is

$$P(x, v, t) = f(x - vt, v), \tag{1.7}$$

where $f(x, v)$ is an arbitrary function of two arguments, defined for all values $-\infty < x, v < \infty$, which represents the initial state

$$P(x, v, 0) = f(x, v). \tag{1.8}$$

The condition (1.6b) leads to

$$f(x - vt, v) = f(x - vt - 2l, v)$$

and (1.6a) to

$$f(x - vt, v) = f(-x + vt, -v).$$

The first of these conditions says that $f(x, v)$ is periodic in x with the period $2l$,

$$f(x, v) = f(x + 2l, v); \tag{1.9a}$$

the second, that it is symmetric for the inversion

$$f(x, v) = f(-x, -v). \tag{1.9b}$$

These two periodicity conditions define the periodic continuation of $P(x, v, t)$.

The case of a particle having for $t = 0$ almost a fixed position x_0 and fixed velocity v_0 is of particular interest. In order to describe it in a simple way we introduce a function $\varphi(x, v)$ restricted to a narrow domain around $x = 0$, $v = 0$; assuming φ to be normalized, the average of a function $q(x, v)$ is defined by

$$\overline{q} = \int_0^l \int_{-\infty}^{\infty} q(x, v)\, \varphi(x, v)\, dx\, dv, \quad \int_0^l \int_{-\infty}^{\infty} \varphi(x, v)\, dx\, dv = 1, \tag{1.10}$$

and we postulate

$$\overline{x} = 0, \ \overline{v} = 0, \ \overline{x^2} = \sigma_0^2, \ \overline{v^2} = \tau_0^2, \tag{1.11}$$

where $\sigma_0 \ll l$, $\tau_0 \ll v_0$.

Then, the function

$$f(x, v) = \sum_{k=-\infty}^{\infty} \left\{ \varphi(2kl + x - x_0, v - v_0) + \varphi(2kl - x - x_0, -v - v_0) \right\} \tag{1.12}$$

has all properties requested: it satisfies (1.9a) and (1.9b) and it has, in the interval $0 < x < l$, only one sharp maximum corresponding to the first term for $k = 0$ (as the maximum of the second term, at $-x_0 + 2kl$, is outside the interval for all $k = 0, \pm 1, \pm 2, \ldots$).

Hence, the probability density is, according to (1.7),

$$\left.\begin{aligned} & P(x, v, t) \\ = & \sum_{k=-\infty}^{\infty}\{\varphi(2kl + x - x_0 - vt, v - v_0) + \varphi(2kl - x - x_0 + vt, v - v_0)\}; \end{aligned}\right\} \quad (1.13)$$

it is properly normalized, for

$$\left.\begin{aligned} \int_0^l \int_{-\infty}^{\infty} P(x, v, t)\, dx\, dv &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} d\eta \left[\int_{2kl - x_0 - (v_0+\eta)t}^{(2k+1)l - x_0 - (v_0+\eta)t} \varphi(\xi, \eta)\, d\xi - \int_{2kl - x_0 - (v_0+\eta)t}^{(2k-1)l - x_0 - (v_0+\eta)t} \varphi(\xi, \eta)\, d\xi\right] \\ &= \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\xi\, \varphi(\xi, \eta) = 1, \end{aligned}\right\} \quad (1.13\,a)$$

in virtue of (1.10).

If $\varphi(x, v)$ is chosen as a Dirac δ-function, i. e. $\sigma_0 = 0$, $\tau_0 = 0$, this function (1.13) reduces to zero except for the points which satisfy the equations (1.1), (1.2). But this limiting case does not correspond to a real physical situation. We have to consider σ_0 and τ_0 as finite quantities.

By integrating (1.13) over v one obtains the spatial distribution

$$\left.\begin{aligned} & P(x, t) = \int_{-\infty}^{\infty} P(x, v, t)\, dv \\ = & \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \{\varphi(2kl + x - x_0 - (v_0 + \eta)t, \eta) + \varphi(2kl - x - x_0 - (v_0 + \eta)t, \eta)\} d\eta, \end{aligned}\right\} \quad (1.14)$$

and, by integrating (1.13) over x from 0 to l, the velocity distribution

$$P(v, t) = \int_{-\infty}^{\infty} P(x, v, t)\, dx = \sum_{k=-\infty}^{\infty} \left\{\int_{2kl - x_0 - vt}^{(2k+1)l - x_0 - vt} \varphi(\xi, v - v_0)\, d\xi - \int_{2kl - x_0 + vt}^{(2k-1)l\, x_0 + vt} \varphi(\xi, -v - v_0)\, d\xi\right\}. \quad (1.15)$$

These two formulae are the analytical expression of the fact that at each reflection the velocity changes its sign. The distribution of the absolute value of the velocity is obviously nothing but the probability that the velocity is either v or $-v$, hence

$$P(|v|, t) = P(v, t) + P(-v, t). \quad (1.16)$$

This quantity is easily found from (1.15) to be independent of time, as should be expected. For the two parts in (1.16) contribute terms in the sum (1.15) which can be combined to integrals from $-\infty$ to ∞:

$$P(|v|, t) = P(|v|) = \int_{-\infty}^{\infty} \{\varphi(\xi, v - v_0) + \varphi(\xi, -v - v_0)\}\, d\xi. \quad (1.17)$$

As an example for which all calculations can be performed in detail, one can consider $\varphi(x, v)$ as a Gauss function in both arguments. If we put

$$\varphi(x, v) = \frac{1}{2\pi\sigma_0\tau_0} e^{-\frac{x^2}{2\sigma_0^2} - \frac{v^2}{2\tau_0^2}}, \quad (1.18)$$

the equations (1.11) are satisfied. (1.13) becomes

$$\left.\begin{aligned} P(x, v, t) = \frac{1}{2\pi\sigma_0\tau_0} \Bigg\{ & e^{-\frac{(v-v_0)^2}{2\tau_0^2}} \sum_{k=-\infty}^{\infty} e^{-\frac{1}{2\sigma_0^2}(2kl + x - x_0 - vt)^2} \\ & + e^{-\frac{(v+v_0)^2}{2\tau_0^2}} \sum_{k=-\infty}^{\infty} e^{-\frac{1}{2\sigma_0^2}(2kl - x - x_0 - vt)^2} \Bigg\}, \end{aligned}\right\} \quad (1.19)$$

and (1.14)

$$P(x, t) = \frac{1}{\sigma(t)\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left\{ e^{-\frac{1}{2\sigma(t)^2}(2kl + x - x_0 - v_0 t)^2} + e^{-\frac{1}{2\sigma(t)^2}(2kl - x - x_0 - v_0 t)^2} \right\}, (1.20)$$

where

$$\sigma(t) = \sqrt{\sigma_0^2 + \tau_0^2 t^2}. \quad (1.21)$$

If now the averages of x, x^2, and $(\Delta x)^2 = (x - \bar{x})^2$ are formed with the distribution (1.20) one finds for $\bar{x}$ exactly the expressions (1.1), (1.2) and further

$$\overline{(\Delta x)^2} = \sigma(t)^2. \quad (1.22)$$

Hence, the width of the distribution increases with time. It becomes equal to the whole range l of x at a critical instant

$$t_c = \frac{1}{\tau_0}\sqrt{l^2 - \sigma_0^2}. \quad (1.23)$$

If $\sigma_0 \ll l$, this is approximately $t_c \sim l/\tau_0$, the value used in Part I. For small τ_0, the epoch t_c is very large but always finite.

It can now be shown that, for $t \to \infty$, $P(x, t)$ becomes constant, independent of x and t. If t is large, one has $\sigma(t) \to \tau_0 t$, and (1.20) reduces to

$$P(x, t) \to \frac{1}{\tau_0 t \sqrt{2\pi}} \, 2 \sum_{k=-\infty}^{\infty} e^{-\frac{1}{2\tau_0^2}\left(\frac{2lk}{t} - v_0\right)^2};$$

if one puts $\frac{2lk}{t} - v_0 = \eta$, then to an increment $\Delta k = 1$ there corresponds $\Delta\eta = 2l/t$ which, for $t \to \infty$, tends to zero. Hence, the sum goes over into an integral

$$P(x, t) \to \frac{1}{\tau_0 \sqrt{2\pi}} \frac{1}{l} \int_{-\infty}^{\infty} e^{-\frac{\eta^2}{2\tau_0^2}} d\eta = \frac{1}{l}. \tag{1.24}$$

This is the properly normalized "geometrical" probability for finding the particle anywhere in the interval of length l.

However, the distribution for $t \to \infty$ is not that of an ideal gas, as the velocity distribution is different. One obtains from (1.17) and (1.18)

$$P(|v|) \to \frac{1}{\tau_0 \sqrt{2\pi}} \left\{ e^{-\frac{(v - v_0^2)}{2\tau_0^2}} + e^{-\frac{(v + v_0)^2}{2\tau_0^2}} \right\}, \tag{1.25}$$

that means two Gauss distributions with the mean velocities $\pm v_0$, but not a Maxwell distribution.

The result of this consideration is therefore that a motion which starts as that of a practically individualistic particle, in the course of time goes over into a state where the position becomes completely indetermined while the magnitude of the velocity remains unchanged, its direction indetermined.

The question how the model has to be modified so that the final state is an ideal gas will not be investigated here in detail. It is obvious that a mechanism for the exchange of velocities between several mobile objects is needed. I presume that it suffices to replace one of the elastic boundaries by a model of a thermal reservoir (a heavy body with a Maxwell energy distribution) which exchanges energy and momentum with the particle at each collision.

2. Quantum mechanics of the one-dimensional one-particle gas.

To treat the same problem with quantum mechanics one has to solve the time-dependent Schrödinger equation for the wave function $\psi(x, t)$,

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\hbar i\frac{\partial\psi}{\partial t}=0 \tag{2.1}$$

with the boundary conditions

$$\psi(0, t)=0, \quad \psi(l, t)=0. \tag{2.2}$$

There are two standard methods, that of d'Alembert and that of Fourier. The d'Alembertian solution is, in the present case, preferable as it leads to results easily comparable with those of the classical treatment. The transformation in a Fourier series can then be easily obtained.

De Broglie has given, in one of his books (6), a solution[1] of (2.1) without boundaries, which corresponds to arbitrary initial values $f(x)$; namely

$$\psi(x, t)=\left(\frac{-im}{2\pi\hbar t}\right)^{1/2}\int_{-\infty}^{\infty} f(\xi)e^{\frac{im}{2\hbar t}(x-\xi)^2}d\xi. \tag{2.3}$$

This can be readily confirmed by direct calculation (substituting into (2.1) and demonstrating that $\psi(x, t)\rightarrow f(x)$ for $t\rightarrow 0$). Then, following Darwin, he choses for $f(x)$ the function

$$f(x)=\left(\frac{1}{\sigma_0\sqrt{2\pi}}\right)^{1/2}e^{-\frac{(x-x_0)^2}{4\sigma_0^2}+\frac{i}{\hbar}mv_0(x-x_0)}, \tag{2.4}$$

which represents an harmonic wave with momentum mv_0, modulated by a Gauss function with a crest at x_0 and width $\sigma_0\sqrt{2}$. The probability for location $|f(x)|^2$ is normalized,

$$\int_{-\infty}^{\infty} f|(x)|^2\,dx=1, \tag{2.5}$$

[1] De Broglie has actually treated the three-dimensional case.

and the expectation values of coordinate, momentum, and their mean square deviations are

$$\left.\begin{aligned} \overline{x} &= \int_{-\infty}^{\infty} x\, ff^*\, dx = x_0, \\ \overline{p} &= -\hbar i \int_{-\infty}^{\infty} f^* \frac{df}{dx}\, dx = m v_0; \end{aligned}\right\} \tag{2.6a}$$

$$\left.\begin{aligned} \overline{(\varDelta x)^2} &= \int_{-\infty}^{\infty} x^2 ff^*\, dx - x_0^2 = \sigma_0^2, \\ \overline{(\varDelta p)^2} &= -\hbar^2 \int_{-\infty}^{\infty} f^* \frac{d^2 f}{dx^2}\, dx - m^2 v_0^2 = \frac{\hbar^2}{4\,\sigma_0^2}. \end{aligned}\right\} \tag{2.6b}$$

If we introduce the uncertainty of the velocity

$$\tau_0 = \sqrt{\overline{(\varDelta v)^2}} = \frac{1}{m}\sqrt{\overline{(\varDelta p)^2}} = \frac{\hbar}{2\,\sigma_0 m}, \tag{2.7}$$

we have the Heisenberg uncertainty relation

$$\sqrt{\overline{(\varDelta x)^2}\cdot\overline{(\varDelta p)^2}} = m\,\sigma_0 \tau_0 = \frac{\hbar}{2}. \tag{2.8}$$

If (2.4) is substituted in (2.3) and the integration performed, one obtains after some reduction

$$\left.\begin{aligned} \psi(x,t) = \left(\frac{s(t)}{\sigma(t)\sqrt{2\pi}}\right)^{1/2} exp\Bigg\{ &-\left(\frac{x-x_0-v_0 t}{2\,\sigma(t)}\right)^2 \\ &-\frac{im}{2\,\hbar\, t}\left[\frac{\sigma_0^2}{\sigma(t)^2}(x-x_0-v_0 t)^2-(x-x_0)^2\right]\Bigg\}_1, \end{aligned}\right\} \tag{2.9}$$

where

$$s(t) = \frac{\sigma_0 - i\tau_0 t}{\sigma(t)}, \qquad |s(t)|^2 = 1. \tag{2.10}$$

$\psi(x, t)$ is the normalized probability amplitude for a group of waves with a crest initially at x_0 moving with the velocity v_0 (from left to right). Then, $\psi(-x, t)$ corresponds to a group of waves with a crest initially at $-x_0$ and moving with the velocity

$-v_0$ (from right to left). For inspection of (2.9) shows that a change of sign of x is equivalent to a change of signs of x_0 and v_0.

Applying the image method, we construct the function

$$\Psi(x, t) = \sum_{k=-\infty}^{\infty} \{\psi(2kl + x, t) - \psi(2kl - x, t)\}; \quad (2.11)$$

it is obviously periodic in x with period $2l$ and vanishes for $x = 0$ and $x = l$. If $t \to 0$, one has approximately $\sigma(t) \to \sigma_0$, $s(t) \to 1$, and

$$\left.\begin{aligned}\Psi(x, 0) = \left(\frac{1}{\sigma_0 \sqrt{2\pi}}\right)^{1/2} \sum_{k=-\infty}^{\infty} \Bigg\{ & exp\left[-\left(\frac{2kl + x - x_0}{2\sigma_0}\right)^2 + \frac{imv_0}{\hbar}(2kl + x - x_0)^2\right] \\ & - exp\left[-\left(\frac{2kl - x - x_0}{2\sigma_0}\right)^2 + \frac{imv_0}{\hbar}(2kl - x - x_0)^2\right]\Bigg\}.\end{aligned}\right\} \quad (2.12)$$

This can be written

$$\Psi(x, 0) = \sum_{k=-\infty}^{\infty} \{f(2kl + x) - f(2kl - x)\}, \quad (2.13)$$

where $f(x)$ is the function defined by (2.4). Hence, the initial state consists in two groups of plane waves travelling to the right and left, both modulated by Gauss functions of width σ_0, and group crests at $x_0 + 2kl$ and $-x_0 + 2kl$ ($k = 0, \pm 1, \pm 2, \cdots$), respectively. Inside the interval $0 \leqslant x \leqslant l$, these waves are equivalent to one wave with a crest initially at x_0, which is repeatedly reflected at the boundaries $x = 0$ and $x = l$. Hence the solution describes, for small σ_0, a repeatedly reflected single particle with slightly uncertain initial position.

The probability of location is

$$\left.\begin{aligned}P(x, t) = \Psi\Psi^* = \sum_{k=-\infty}^{\infty} \sum_{k^1=-\infty}^{\infty} & \{\psi(2k'l + x, t) \\ & - \psi(2kl - x, t)\}\{\psi^*(2kl + x, t) - \psi^*(2kl - x, t)\}.\end{aligned}\right\} \quad (2.14)$$

Now, each term $\psi(2kl + x, t)$ corresponds, in Fig. 1, to a line ascending from left to right (+ line), each term $\psi(2kl - x, t)$ to a line ascending from right to left (— line). Accordingly, the four products obtained by multiplying out the bracket in (2.14) can be classified into three types and the total probability split into three parts:

$$P(x, t) = P_c(x, t) + P_i(x, t) + P_r(x, t). \tag{2.15}$$

For $k = k'$, the terms $\psi(2kl + x, t)\, \psi^*(2kl + x, t)$ and $\psi(2kl - x, t)\, \psi^*(2kl - x, t)$ represent the superposition of the Gauss function of a (+ line) with itself and a (— line) with itself; they contribute to (2.15)

$$P_c(x, t) = \frac{1}{\sigma(t)\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left\{ e^{-\frac{1}{2\sigma(t)^2}(2kl + x - x_0 - v_0 t)^2} + e^{-\frac{1}{2\sigma(t)^2}(2kl - x - x_0 - v_0 t)^2} \right\}, \tag{2.16}$$

which is identical with the probability (1.20) derived from the classical theory.

The remaining terms for $k = k'$, namely $-\psi(2kl + x, t)\, \psi^*(2kl - x, t)$ and $-\psi(2kl - x, t)\, \psi^*(2kl + x, t)$, correspond each to the intersection point of a (+ line) with an equally numbered (— line); all these are (cf. Fig. 1) on the boundary $x = 0$. It is obvious that the other boundary $x = l$, where $k' = k + 1$, contributes terms of the same type and similar magnitude. Collecting all these terms, we obtain

$$\left.\begin{aligned} P_i(x, t) = \frac{-2}{\sigma(t)\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \Bigg\{ & e^{-\frac{1}{2\sigma(t)^2}[x^2 + (2kl - x_0 - v_0 t)^2]} \\ & \cos\frac{x}{\sigma_0 \tau_0 t}\left[\frac{\sigma_0^2}{\sigma(t)^2}(2kl - x_0 - v_0 t) - 2kl + x_0\right] \\ + e^{-\frac{1}{2\sigma(t)^2}[(l-x)^2 + (2kl - x_0 - v_0 t)^2]} & \cos\frac{l - x}{\sigma_0 \tau_0 t}\left[\frac{\sigma_0^2}{\sigma(t)^2}(2kl - x_0 - v_0 t) - 2kl + x_0\right]\Bigg\}. \end{aligned}\right\} \tag{2.17}$$

These terms represent interference effects due to the superposition of an incident with a reflected wave near one of the boundaries. The fringes, described by the cos-terms, are restricted, by the Gauss functions, to a neighbourhood of the boundary of width $\sigma(t)$; if $\sigma_0 \ll l$, these regions of interference remain narrow for a long time ($t \ll t_c$). The remaining terms, all of the type

$k \neq k'$, correspond either to the superposition of two different (+ lines) or two different (— lines) or to intersection points of a (+ line) and a (—line); outside the region $0 \leqslant x \leqslant l$. If $\sigma(t) \ll l$, their contribution to the probability, $P_r(x, t)$, is small and can be neglected for $t \ll t_c$.

The essential differences between the classical and quantum treatment are now clearly seen to be of two different kinds; there are, firstly, the interference effects near the boundaries, represented by P_i, and, secondly, the Heisenberg uncertainty relation which connects σ_0 and $\tau_0 (= \hbar/2\, m\sigma_0)$ and thus prohibits simultaneously sharp initial position and velocity. Both effects are appreciable only for atomistic particles and negligible for macrobodies (m large).

It is now clear that whenever the interference terms P_i can be neglected, namely when $\sigma_0 \ll l$ and $\tau_0 = \dfrac{\hbar}{2\, m\, \sigma_0} \ll v_0$, or, when

$$\frac{\hbar}{2\, m v_0} \ll \sigma_0 \ll l,$$

then $P(x, t)$ approaches, for $t \to \infty$, the constant value $1/l$ as in (1. 24).

We have now to investigate the relation of the solution for an individual particle given above and the solution based on eigenstates (which Einstein uses for his critical considerations). For this purpose we expand the function $\Psi(x, t)$ in a Fourier series; as it is antisymmetric we can write

$$\Psi(x, t) = \sum_{n=1}^{\infty} A_n(t) \sin \frac{n\pi}{l} x, \tag{2.18}$$

with

$$A_n(t) = \frac{2}{l} \int_0^l \Psi(x, t) \sin \frac{n\pi}{l} x \, dx. \tag{2.19}$$

By substituting (2.18) in the differential equation (2.1), one sees that $A_n(t)$ satisfies the equation

$$\hbar i \frac{\partial A_n(t)}{\partial t} - \left(\frac{\hbar n \pi}{l}\right)^2 \frac{1}{2\, m} A_n(t) = 0, \tag{2.20}$$

and as $E = \hbar i\, \partial/\partial t$ is the energy operator, one has

$$A_n(t) = A_n\, e^{iE_n t/\hbar}, \tag{2.21}$$

where

$$E_n = \left(\frac{\pi\hbar\, n}{l}\right)^2 \frac{1}{2\, m} \tag{2.22}$$

are the eigenvalues of the energy. Therefore it suffices to calculate the constants

$$A_n = A_n(0) = \frac{2}{l}\int_0^l \Psi(x,\, 0)\, \sin\frac{n\,\pi\, x}{l}\, dx. \tag{2.23}$$

Substituting (2.13) one has

$$\begin{aligned} A_n &= \frac{2}{l}\sum_{k=-\infty}^{\infty}\left\{\int_0^l f(2kl + x)\ \sin\frac{n\,\pi\, x}{l}\, dx - \int_0^l f(2kl - x)\ \sin\frac{n\,\pi\, x}{l}\, dx\right\} \\ &= \frac{2}{l}\sum_{k=-\infty}^{\infty}\int_{2k-1}^{2k+1} f(x)\ \sin\frac{n\,\pi\, x}{l}\, dx, \end{aligned}$$

hence

$$A_n = \frac{2}{l}\int_{-\infty}^{\infty} f(x)\ \sin\frac{n\,\pi\, x}{l}\, dx, \tag{2.24}$$

which shows that the Fourier coefficient of $\Psi(x,\, 0)$ in the interval $0 \leqslant x \leqslant l$ is the Fourier transform of $f(x)$ in $-\infty < x < \infty$ taken at the points $n\pi/l$ of the reciprocal space.

It follows now readily that $\Psi(x,\, t)$ is normalized for all t; one has

$$\left.\begin{aligned} \int_0^l \Psi\Psi^*\, dx &= \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty} A_n A_{n'}^*\ e^{i(E_n - E_{n'})\, t/\hbar}\int_0^l \sin\frac{\pi\, nx}{l}\sin\frac{\pi\, n' x}{l}\, dx \\ &= \frac{l}{2}\sum_{n=1}^{\infty}|\, A_n\, |^2 = \int_{-\infty}^{\infty}|f(x)|^2\, dx = 1. \end{aligned}\right\} \tag{2.25}$$

Introducing for $f(x)$ in (2.24) the expression (2.4), one obtains

$$A_n = \frac{\sqrt{2}}{il}\,(\sigma_0\sqrt{2\,\pi})^{1/2}\left\{e^{-\left(\frac{v_0}{2\tau_0} + \frac{\pi\sigma_0 n}{l}\right)^2 + i\frac{\pi nx}{l}} - e^{-\left(\frac{v_0}{2\tau_0} - \frac{\pi\sigma_0 n}{l}\right)^2 - i\frac{\pi nx}{l}}\right\}. \tag{2.26}$$

The absolute value of the momentum in the state n is, according to (2.22),

$$p_n = \sqrt{2\,mE_n} = \frac{\pi\,\hbar\,n}{l}; \tag{2.27}$$

hence, with $\sigma_0\tau_0 = \hbar/2\,m$, (2.26) can also be written

$$A_n = \frac{\sqrt{2}}{il}(\sigma_0\sqrt{2\,\pi})^{1/2}\left\{e^{-\frac{1}{4\tau_0^2}(v_0+p_n/m)^2+ix_0p_n/\hbar} - e^{-\frac{1}{4\tau_0^2}(v_0-p_n/m)^2-ix_0p_n/\hbar}\right\}. \tag{2.28}$$

Assume $v_0 > 0$; since in (2.18) $n = 1, 2, \ldots\ldots, p_n$ is positive. Hence only the exponent of the second term can approach zero, namely for

$$p_n \sim mv_0,\ n_{\max} \sim \frac{mv_0 l}{\hbar\pi},\ E_{\max} \sim \frac{mv_0^2}{2}. \tag{2.29}$$

For this n one has

$$A_{max} \sim i\,\frac{\sqrt{2}}{l}(\sigma_0\sqrt{2\,\pi})^{1/2}\,e^{-imv_0x_0/\hbar}, \tag{2.30}$$

and the expansion (2.18) reduces, for small τ_0, to the leading term:

$$\left.\begin{aligned} \Psi(x,t) &\sim i\,\frac{\sqrt{2}}{l}(\sigma_0\sqrt{2\,\pi})^{1/2}\,e^{-imv_0x_0/\hbar}\sin\frac{mv_0x}{\hbar} \\ &= \frac{1}{\sqrt{2}\,l}(\sigma_0\sqrt{2\,\pi})^{1/2}\left\{e^{i\frac{mv_0}{\hbar}(x-x_0)} - e^{-i\frac{mv_0}{\hbar}(x+x_0)}\right\}. \end{aligned}\right\} \tag{2.31}$$

This is the solution of the Schrödinger equation used by EINSTEIN (cf. Part I, (1)) to demonstrate the incompleteness of quantum mechanics. However, as the preceding considerations show, it is only an approximation; the correct solution is the wave packet with the coefficients (2.26) or (2.28), and this is completely equivalent to the d'Alembertian solution (2.11) which exhibits the fact that, for a restricted time ($t < t_c$), the motion is properly approximated by the classical, orbital or individualistic description. The quantum formula (2.31) and the classical formula (1.1) are therefore bridged by a continuous transition, and there is no paradoxial situation for macro-bodies which EINSTEIN believes to exist.

Einstein's objections against quantum mechanics based on the interference of probabilities can also be illuminated by this model. The first point is that one must not add phase factors of the form $e^{i\alpha_k}$ to the terms of the sum (2.11), because then the boundary (periodicity) conditions would be violated. All the different terms in the sum are in phase; only a common phase factor $e^{i\alpha}$ can be added to the whole sum. But this cancels in the probability expression (2.14). Hence, the interference term given in (2.17) is genuine and cannot be destroyed by averaging over phases; these interferences between incident and reflected wave are of the same type as those in certain interferometric optical experiments (standing waves).

But one can now consider the case, discussed at the the end of Part I, where the initial distribution has two sharp maxima, one at x_1, the other at x_2; i. e. one knows only that the particle is either near x_1 or near x_2. The solution $\Psi(x, t)$ is then a linear combination of the two single functions with complex factors; but the relative phase of these is indetermined, one has to average over it and thus no interference phenomenon results from this situation. This must be so; for simple ignorance where a particle is at $t = 0$ cannot produce a physical interference phenomenon. Observable interference can be obtained only by feeding in particles from one source at two places by a physical instrument which divides one de Broglie wave into two "coherent" beams in a similar way as half-silvered plates and similar devices in optics. As soon as an attempt is made to decide on which of the two feeding branches the particle appears, there is a new initial state and no interference is observable.

3. Summary.

It is misleading to compare quantum mechanics with deterministically formulated classical mechanics; instead, one should first reformulate the classical theory, even for a single particle, in an indeterministic, statistical manner. Then some of the distinctions between the two theories disappear, others emerge with great clarity. Amongst the first is the feature of quantum mechanics, that each measurement interrupts the automatic flow of events and introduces new initial conditions (so-called "reduction of probability"); this is true just as well for a statistically formulated classical theory. The essential quantum effects are of two kinds: the reciprocal relation between the maximum of sharpness for coordinate and velocity in the initial and consequently in any later state (uncertainty relations), and the interference of probabilities whenever two (coherent) branches of the probability function overlap. For macro-bodies both these effects can be made small in the beginning and then remain small for a long time; during this period the individualistic description of traditional classical mechanics is a good approximation. But there is always a critical moment t_c where this ceases to be true and the quasi-individual is transforming itself into a genuine statistical ensemble.

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